## (2) Introduction to Dimensional Analysis ${ }^{2}$

Usually, a certain physical quantity, say, length or mass, is expressed by a number indicating how many times it is as large as a certain unit quantity. Therefore, the statement that the length of this stick is 3 does not make sense; we must say, with a certain unit, for example, that the length of this stick is 3 m . A number with a unit is a meaningless number as the number itself ( 3 m and $9.8425 \cdots$ feet are the same). That is, we may freely scale it through choosing an appropriate unit. In contrast, the statement that the ratio of the lengths of this and that stick is 4 makes sense independent of the choice of the unit. A quantity whose numerical value does not depend on the choice of units is calle a dimensionless quantity. The number 4 here is dimensionless, and has an absolute meaning in contrast to the previous number 3. The statement that the length of this stick is 3 m depends not only on the property of the stick but also on how we observe (or describe) it. In contrast, the statement that the ratio of the lengths is 4 is independent of the way we describe it.

A formula describing a relation among several physical quantities is actually a relation among several numbers. If the physical relation holds 'apart from us,' or in other words, is independent of the way we describe it, then whether the relation holds or not should not depend on the choice of the units or such 'convenience to us.' A relation correct only when the length is measured in meters is hardly a good objective relation among physical quantities (it misses an important universal property of the law of physics, or at least very inconvenient).

When we switch units, the accompanying numbers are scaled. However, the numbers with the same unit must be scaled in an identical way. Since there are many physical quantities with different units such as length and quantity of electric charge, physical quantities with distinct units may be scaled independently when we switch units. If two quantities scale always identically when we switch units arbitrarily, we say these two quantities have the same dimension. In other words scientists and engineers express independent scalability as having different dimensions. An objective relation among physical quantities must keep holding even when we scale all the quantities with different dimensions independently. Therefore, such a relation must be expressed solely in terms of dimensionless quantities (= scaling invariant quantities). To analyze a problem using this requirement is called dimensional analysis.

Let us look at a simple example. Consider a 1 dimensional diffusion equation

$$
\begin{equation*}
\frac{\partial \psi}{\partial t}=D \frac{\partial^{2} \psi}{\partial x^{2}}, \tag{12}
\end{equation*}
$$

where $t$ is time, $x$ is the spatial coordinate along the $x$-axis, $\psi$ is, say the temperature, and $D$ is the diffusion constant. Let us write the total amount obtained by spatially integrating $\psi(x, t)$ as $Q(t)$.

There are three distinct units of physical quantities: time, distance (= length) and the quantity of something diffusing (the unit of $Q$ ). Let us denote the dimensions of these unit quantities as $T, L$ and $M$, respectively. It is customary to write the dimension of a quantity $X$ as $[X]$. Thus, $[t]=T,[x]=L$, and $[Q]=M . \psi$ is the density of the diffusing 'substance'

[^0](in the 1 D world) it is $[\psi]=M / L$. Differentiation is nothing but division dimensionally, $[\partial \psi / \partial t]=M / L T$, for example. The both sides of (12) must have the same dimensions; we say the formula is dimensionally homogeneous). This means that the equation continues to hold, even if we change the unit of length from, say $m$ to inch. That is, no inconsistency emerges with arbitrary changes of units. We do not lose any generality assuming only dimensionally homogeneous relations are meaningful in physics (or physically meaningful equations can always be rewritten in the homogeneous form).

To perform the dimensional analysis of the diffusion equation (12), we must first construct dimensionless quantities. We make combinations of quantities for which all the powers of $T, L, M$ are zero: $t D / x^{2}$ and $\psi x / Q$ or $\psi \sqrt{t D} / Q$ are dimensionless. ${ }^{3}$ For example, $\left[t D / x^{2}\right]=T \cdot\left(L^{2} / T\right) / L^{2}=1$. A dimensionless quantity must be a function of dimensionless quantities only (think what happens if this is not true), so a solution to (12) must have the following form:

$$
\begin{equation*}
\psi=\frac{Q}{\sqrt{D t}} f\left(\frac{x^{2}}{D t}\right) \tag{13}
\end{equation*}
$$

where $f$ is a well-behaved function. ${ }^{4}$ Putting this form into (12), we obtain an ordinary differential equation for $f$, which is much easier to solve than the original PDE.

A remark is in order here. Whether we may regard two physical quantities with different dimensions or not can be a problem. For example, in the usual engineering energy and mass have distinct units (so with distinct dimensions). It is natural, however, to regard them to have the same dimension in relativity where mass and energy convert into each other. The famous formula $E=m c^{2}$ contains a 'conversion factor' $c^{2}$ simply due to the non-relativistic custom. Relativistically, the speed of light $c$ is a universal constant independent of any observer, so it is much more natural to regard $c^{2}$ as a dimensionless parameter. Consequently, length and time must have the same dimension. However, needless to say, in our usual nonrelativistic world, $c$ is so large that it is hardly distinguished from $\infty$. Therefore, it drops out from the formula and space and time are safely and conveniently regarded categorically distinct.

Dimensional analysis can often be crucial. For example, we can understand why atoms cannot be understood classical physically. A hydrogen atom consists of an electron trapped by a proton with the Coulomb interaction, so its Newton's equation of motion reads

$$
\begin{equation*}
m \frac{d^{2}}{d t^{2}} \boldsymbol{r}=-\frac{e^{2} \boldsymbol{r}}{4 \pi \epsilon_{0} r^{3}}, \tag{14}
\end{equation*}
$$

where $e$ is the elemental charge (the charge of proton) and $m$ is the mass of the electron. $4 \pi \epsilon_{0}$ always appears with $e^{2}$, so the fundamental variables and parameters are only two: $m$ and $e^{2} / 4 \pi \epsilon_{0} .[m]=M$. Since $\left[e^{2} / 4 \pi \epsilon_{0}\right]=M L^{3} / T^{2}$ (this you can see from the homogeneity

[^1]of the equation), there is no way to construct a quantity with the dimension of length $L$. Bohr thought, however, Planck's constant $h$ must be relevant in the atomic world. Since $[h]=M L^{2} / T$ (recall that $h$ times the frequency is the energy of a photon),
\[

$$
\begin{equation*}
\left[e^{2} / m \epsilon_{0}\right]=L^{3} T^{-2}, \quad[h / m]=L^{2} T^{-1} . \tag{15}
\end{equation*}
$$

\]

From these we may solve $L .(h / m)^{2} /\left(e^{2} / m \epsilon_{0}\right)$ is the answer:

$$
\begin{equation*}
\left[(h / m)^{2} /\left(e^{2} / m \epsilon_{0}\right)\right]=\left(L^{2} T^{-1}\right)^{2} /\left(L^{3} T^{-2}\right)=L . \tag{16}
\end{equation*}
$$

$\epsilon_{0} h^{2} / m e^{2} \times 1 / \pi$ is Bohr's radius $0.53 \AA=0.053 \mathrm{~nm}$ (nanometer, $1 \mathrm{~nm}=10^{-9} \mathrm{~m}$ ) is an approximate size of the hydrogen atom in its ground state. Actually, Bohr used a dimensional analytic argument to convince himself that $h$ was the key.

Quiz 3.5A. 1 Derive Kepler's third law dimensional analytically.
Quiz 3.5A. 2 Migdal's interesting book ${ }^{5}$ begins with a dimensional analytical proof of Pythagorus' theorem. This is based on the argument that the area $S$ of an orthogonal triangle with the smallest angle $\alpha$ and the length of the hypotenuse $a$ may be written as $S=a^{2} f(\alpha)$. As can be seen from the figure $a^{2}+b^{2}=c^{2}$. Is this really a respectable proof?

Also think what could happen, if the space was curved.


Pythgorus' theorem. The argument does not contradict a consequence of the Euclidean axiomatic system, but how can we demonstrate $S=a^{2} f(\alpha)$ from the system? Is Migdal really logical?

[^2]
[^0]:    ${ }^{2}$ Appendix 3.5B of Y. Oono, Nonlinear World (U Tokyo Press, Jan 2009 to appear).

[^1]:    ${ }^{3}$ There is no other independent combination. That is, all the dimensionless quantities are written in terms of the product of appropriate powers of these two dimensionless quantities. A general theorem relevant to this is called Bridgemen's $\Pi$ theorem, but we need not such general discussion. In this example, there are three (3) independent dimensions $T, L, M$. There are five (5) variables and parameters $Q, \psi, D, t, x$. Therefore, there are $5-3=2$ independent dimensionless quantities.
    ${ }^{4}$ The meaning of the word 'well-behaved' is context dependent, but usually, it means needed differentiability, boundedness in the domain under consideration, etc.

[^2]:    ${ }^{5}$ A. B. Migdal, Qualitative methods in quantum theory (translated by A. J. Leggett).

