

Introduction to Applicable Analysis

Yoshi Oono 1997 Spring Version

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- 41B.8 Universal Turing machine

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41C.9 Theorem

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A1 Point set and limit

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A1.2 Convergence, limit

A1.3 Theorem [Cauchy]

A1.4 Symbol ' O ' and ' o '

A1.5 Limit and arithmetic operations commute

A1.6 Lower and upper bound, supremum and infimum

A1.7 Monotone sequences

A1.8 Theorem [Bounded monotone sequences converge]

A1.9 Divergence to \pm infinity

A1.10 Limsup and liminf

A1.11 Infinite series

A1.12 Absolute convergence

A1.13 Power series

A1.14 Conditional convergence, alternating series

A1.15 Theorem [Nested sequence of intervals shrinking to a point share the point]

A1.16 Denumerability

A1.17 Cantor's theorem [Continuum is not denumerable]

A1.18 n -space, distance, ϵ -neighborhood

A1.19 Inner point, boundary, accumulation point, closure, open kernel

A1.20 Open set, closed set

A1.21 Limit of point sequence

A1.22 Bounded set, diameter

A1.23 Theorem [Shrinking nested sequence of bounded closed sets]

A1.24 Covering

A1.25 Compact set

A1.26 Theorem [Compactness is equivalent to bounded closedness]

A1.27 Theorem [Bolzano and Weierstrass]

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A2.2 Limit of function

A2.3 Cauchy's criterion

- A2.4 Graph of a function
- A2.5 Continuity
- A2.6 Left and right continuity
- A2.7 Theorem of middle value
- A2.8 Uniform continuity
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- A2.10 Composite function
- A2.11 Monotone function
- A2.12 Inverse function
- A2.13 Even and odd function

A3 Differentiation

- A3.1 Differentiability, derivative
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- A3.3 Increment, differential quotient
- A3.4 Right or left differentiable
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- A3.6 Derivative of composite function
- A3.7 Derivative of inverse function
- A3.8 Theorem [Mean-value theorem]
- A3.9 Theorem [Rolle's theorem]
- A3.10 Theorem [Generalization of mean-value theorem]
- A3.11 Theorem [Condition for monotonicity]
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- A3.18 Theorem [Convexity and second derivative]
- A3.19 Local maximum, minimum
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- A3.23 Theorem [Existence of mollifier]
- A3.24 Theorem [Identity theorem]
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A4 Integration

- A4.1 Definite integral (Riemann integral)
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- A4.3 Basic properties of definite integral
- A4.4 Theorem [Mean value theorem]
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- A5.4 Theorem [Comparison theorem II. comparison of series]
- A5.5 Cauchy's convergence criterion
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- A5.7 Abel's formula
- A5.8 Function sequence, convergence
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- A5.11 Function series, convergence, uniform convergence, maximall convergence
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- A5.13 Theorem [Dini's theorem]
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- A5.17 Theorem [Arzela's theorem]
- A5.18 Majorant
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- A5.21 Theorem [Power series defines a real analytic function]
- A5.22 Theorem [Continuity at $x = r$ or $-r$]
- A5.23 Infinite product
- A5.24 Convergence of infinite product
- A5.25 Theorem Convergence condition for infinite product]
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- A6.2 Function, domain, range
- A6.3 Limit
- A6.4 Continuity
- A6.5 Theorem [Maximum value theorem]
- A6.6 Partial differentiation
- A6.7 Differentiability, total differential
- A6.8 Theorem [Partial differentiability and differentiability]
- A6.9 Theorem [Order of partial differentiation]
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- A6.14 Composite function
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- A7.10 Piecewise C^1 -function
- A7.11 Theorem

- A7.12 Theorem
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- A7.14 Fourier transform
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- A8.6 Lipschitz condition
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- A8.8 Theorem
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- A8.11 Perfect differential equation, integrating factor
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- A8.16 Inhomogeneous equation, Lagrange's method of variation of constants

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- A9.1 Gradient
- A9.2 Coordinate expression of $\text{grad} f$
- A9.3 Remark
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- A9.5 Divergence
- A9.6 Cartesian expression of div
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A9.9 Cartesian expression of curl

A9.10 Potential field. potential, solenoidal field, irrotational field

A9.11 Some formulas

A9.12 Theorem [Gauss-Stokes-Green's theorem]

A9.13 Laplacian

A9.14 Laplacian for vector field