Statistical Physics of Spin Glasses and Information Processing

H Nishimori
Oxford 2001

1. Mean-Field Theory of Phase Transitions

Mean field Theory:
5 Mean field Hamiltonian: In terms of spin fluctuation δS_i the Ising Hamiltonian can be rewritten as

$$\begin{aligned} H &= -J\sum_{\langle i,j\rangle}(m+\delta S_i)(m+\delta S_j) - h\sum S_i, && (1)\\ &\sim -Jm^2N_B - Jm\sum(\delta S_i + \delta S_j) - h\sum S_i, && (2)\\ &= -Jm^2N_B - Jmz\sum \delta S_i - h\sum S_i, && (3)\end{aligned}$$

where N_B is the number of bonds, and z is the coordination number ($N_B = zN/2$). The third line is due to the number of times δS_i appears in the sum (number of bonds connected to j). Replacing δS with $S - m$, we get

$$H = N_B J m^2 - (Jmz = +h)]\sum S_i. \tag{4}$$

6 The free energy may be obtained as

$$F = -Nk_BT\log\{2\cosh\beta(Jmz+h)\} + N_BJm^2 \simeq -Nk_BT\log 2 + \frac{JzN}{2}(1-\beta Jz)m^2 + \frac{N}{12}(Jzm)^4\beta^3. \tag{5}$$

This is the starting point of the Landau theory.

7 Infinite-range model:
Its Hamiltonian is given by

$$H = -\frac{J}{2N}\sum_{i\neq j} S_jS_j - h\sum S_i = -\frac{J}{2N}\left(\sum S_i\right)^2 + \frac{J}{2} - h\sum S_i, \tag{6}$$

8 where the second term came from $\sum S_i^2 = N$. This term is of $O[1]$, so we may ignore it. Using the Gaussian trick, we compute the partition function as

$$\begin{aligned} Z &= Tr\exp\left(\frac{\beta J}{2N}\left(\sum S_i\right)^2 + \beta h\sum S_i\right), && (7)\\ &= Tr\sqrt{\frac{\beta JN}{2\pi}}\int_{-\infty}^{\infty} dm\,\exp\left(-\frac{\beta Jm^2}{2N} + \beta(Jm+h)\sum S_i\right), && (8)\\ &= Tr\sqrt{\frac{\beta JN}{2\pi}}\int_{-\infty}^{\infty} dm\,\exp\left(-\frac{\beta Jm^2}{2N} + N\log\{2\cosh\beta(Jm+h)\}\right). && (9)\end{aligned}$$

This integral is evaluated by Laplace's method: the peak is at

$$\frac{\partial}{\partial m}\left(-\frac{\beta Jm^2}{2N} + N\log\{2\cosh\beta(Jm+h)\}\right) = 0; \tag{10}$$

or

$$m = \tanh \beta (Jm + h). \tag{11}$$

This is the usual mean field equation with $J \to J/N$ and $z \to N$ replacement. The mean-field theory is the exact solution for the infinite range model. The saddle condition (11) can be written as (see (8))

$$m = \frac{1}{N} \sum S_i. \tag{12}$$

If LLN applies this equality is almost surely correct. That is, in the thermodynamic limit fluctuations disappear and the mean-field theory becomes exact.

9 Variational approach:
The sum of the Ising partition function is difficult, because S_i's couple, and we need a simultaneous distribution. Let us decouple as

$$P(\{S_i\}) = \prod P_i(S_i) \tag{13}$$

and determine P to give the best approximation: the minimum free energy: $F = E - TS$

$$F = -J \sum_{\langle i,j \rangle} TrS_iS_jP_i(S_i)P_j(S_j) - h \sum TrS_iP_i(S_i) + k_BT \sum TrP_i(S_i) \log P_i(S_i). \tag{14}$$

Minimizing this wrt P

$$\frac{\delta F}{\delta P_i(S_i)} = -J \sum_j m_j S_i - hS_i + k_BT \log P_i(S_i) + k_BT + \lambda = 0, \tag{15}$$

where λ is the Lagrange coefficient taking into account the normalization of P. Thus, we obtain

$$P_i(S_i) = \frac{1}{Z_{MF}} \exp \left(\beta J \sum_j S_i m_j + \beta h S_i \right). \tag{16}$$

11 Notice that the mean-field Hamiltonian (4) appears in the exp factor.

For Ising spins $S_i = \pm 1$, we may write

$$P_i(S_i) = \frac{1}{2}(1 + m_i S_i), \tag{17}$$

which is compatible with $m_i = TrS_iP_i(S_i)$. Putting this into (14) we can write F in terms of m_i. Minimizing this wrt m_i we obtain

$$m_i = \tanh \beta \left(J \sum_j m_j + h \right). \tag{18}$$

2. Mean-Field Theory of Spin Glasses

12 Edwards-Anderson model:

$$H = -\sum_{\langle i,j \rangle} J_{ij} S_i S_j, \tag{19}$$

where J_{ij} is an iid obeying $N(J_0, J)$ or $\pm J$ Bernoulli $B(p, 1-p)$. The randomness in the model may come from the random composition. For each system $\{J_{ij}\}$ is given (quenched system) so the free energy observed is obtained by the *configurational average*

$$[F] = -k_B T[\log Z] = -k_B T \int \prod_{ij} P(J_{ij}) \log Z, \tag{20}$$

where Z is the partition function for a given $\{J_{ij}\}$.

13 The free energy per spin $f = F(\{J\})/N$ is almost surely identical to the its average $[f]$ in the thermodynamic limit (*self-averaging property* of the free energy), so we may sue f and $[f]$ interchangeablly. The man is easier to compute.

Sherrington-Kirkpatrick model:
This is an infinite range version of the Edwards-Anderson model. Now, $\{J_{ij}\}$ is a set of iid obeying $N(J_0/N, J/\sqrt{N})$.

14 We use the replica trick.

$$[Z^n] = \left[Tr \exp\left(\beta \sum_{i<j} J_{ij} \sum_{\alpha=i}^{n} S_i^\alpha S_j^\alpha + \beta h \sum_i \sum_\alpha S_i^\alpha \right) \right], \tag{21}$$

where α is the replica index. The average over J can be done as

$$Tr \exp \left\{ \frac{1}{N} \sum_{i<j} \left(\frac{1}{2} \beta^2 J^2 \sum_{\alpha,\beta} S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta + \beta J_0 \sum_\alpha S_i^\alpha S_j^\alpha \right) + \beta h \sum_i \sum_\alpha S_i^\alpha \right\}, \tag{22}$$

This can be rewritten as

$$[Z^n] = \exp\left(\frac{N\beta^2 J^2 n}{4} \right) Tr \exp \left\{ \frac{\beta^2 J^2}{2N} \sum_{\alpha<\beta} \left(\sum_i S_i^\alpha S_j^\beta \right)^2 + \frac{\beta J_0}{2N} \sum_a \left(\sum_i S_i^\alpha \right)^2 + \beta h \sum_i \sum_\alpha S_i^\alpha \right\}. \tag{23}$$

15 This can be decoupled with the aid of the Gaussian integrals as

$$\begin{aligned} [Z^n] &= \exp\left(\frac{N\beta^2 J^2 n}{4} \right) \int \mathcal{D}[q] \int \mathcal{D}[m] \exp\left(-\frac{N\beta^2 J^2}{2} \sum_{\alpha<\beta} q_{\alpha\beta}^2 - \frac{N\beta J_0}{2} \sum_\alpha m_\alpha^2 \right) \times \\ & \times Tr \exp\left(\beta^2 J^2 \sum_{\alpha<\beta} q_{\alpha\beta} \sum_i S_i^\alpha S_i^\beta + \beta \sum_\alpha (J_0 m_\alpha + h) \sum_i S_i^\alpha \right). \end{aligned} \tag{24}$$

We may write

$$\begin{aligned} & Tr \exp\left(\beta^2 J^2 \sum_{\alpha<\beta} q_{\alpha\beta} \sum_i S_i^\alpha S_i^\beta + \beta \sum_\alpha (J_0 m_\alpha + h) \sum_i S_i^\alpha \right) \\ = & \left\langle \exp\left(\beta^2 J^2 \sum_{\alpha<\beta} q_{\alpha\beta} S_i^\alpha S_i^\beta + \beta \sum_\alpha (J_0 m_\alpha + h) S_i^\alpha \right) \right\rangle^N \equiv \langle e^L \rangle^N. \end{aligned} \tag{25}$$

16 We thus have

$$[Z^n] = \exp\left(\frac{N\beta^2 J^2 n}{4}\right)\int \mathcal{D}[q]\int \mathcal{D}[m]\exp\left(-\frac{N\beta^2 J^2}{2}\sum_{\alpha<\beta} q_{\alpha\beta}^2 - \frac{N\beta J_0}{2}\sum_{\alpha} m_\alpha^2 + N\log\langle e^L\rangle\right). \tag{26}$$

We apply Laplace's method

$$\begin{aligned}
[Z^n] &= \exp\left(-\frac{N\beta^2 J^2}{2}\sum_{\alpha<\beta} q_{\alpha\beta}^2 - \frac{\beta J_0}{2n}\sum_{\alpha} m_\alpha^2 + N\log\langle e^L\rangle + \frac{N}{4}\beta^2 J^2 n\right), &(27)\\
&\simeq 1 + Nn\left\{-\frac{\beta^2 J^2}{2n}\sum_{\alpha<\beta} q_{\alpha\beta}^2 - \frac{N\beta J_0}{2}\sum_{\alpha} m_\alpha^2 + \frac{1}{n}\log\langle e^L\rangle + \frac{1}{4}\beta^2 J^2\right\}. &(28)
\end{aligned}$$

Here, we take the thermodynamic limit later. Thus,

$$-\beta[f] = \lim_{n\to 0}\frac{[Z^n]-1}{nN} = \lim_{n\to 0}\left\{-\frac{\beta^2 J^2}{2n}\sum_{\alpha<\beta} q_{\alpha\beta}^2 - \frac{N\beta J_0}{2}\sum_{\alpha} m_\alpha^2 + \frac{1}{n}\log\langle e^L\rangle + \frac{1}{4}\beta^2 J^2\right\}. \tag{29}$$

The minimum position is

$$q_{\alpha\beta} = \frac{1}{\beta^2 J^2}\frac{\partial}{\partial q_{\alpha\beta}}\log\langle e^L\rangle = \langle S^\alpha S^\beta\rangle_L, \tag{30}$$

and

$$m_a = \frac{1}{\beta J_0}\frac{\partial}{\partial m_\alpha}\log\langle e^L\rangle = \langle S^\alpha\rangle_L, \tag{31}$$

Thus q and m become the order parameters. q is the *spin glass order parameter.*

Replication-symmetric solution:
18 If $q_{\alpha\beta}$ and m_α are independent of the replica indices, (29) reads

$$-\beta[f] = \frac{\beta^2 J^2}{4n}(-n(n-1)q^2) - \frac{\beta J_0}{2n}nm^2 + \frac{1}{n}\log\langle e^L\rangle + \frac{1}{4}\beta^2 J^2. \tag{32}$$

We need $\langle e^L\rangle$ that can be computed with the Gaussian trick:

$$\begin{aligned}
\langle e^L\rangle &= \left\langle\sqrt{\frac{\beta^2 J^2 q}{2\pi}}\int dz\,\exp\left(-\frac{\beta^2 J^2 q}{2}z^2 + \beta^2 J^2 q z\sum_{\alpha} S^\alpha - \frac{n}{2}b^2 J^2 q + \beta(J_0 m + h) - \frac{n}{2}b^2 J^2 q\right)\right\rangle,\\
&= 1 + n\langle\log 2\cosh\beta\hat{H}(z)\rangle_z - \frac{n}{2}\beta^2 J^2 q + O[n^2], \qquad (33)
\end{aligned}$$

where z obeys $N(0,1)$ and

$$\hat{H}(z) = J\sqrt{q}z + J_0 m + h. \tag{34}$$

Thus, we have obtained

$$-\beta[f] = \frac{\beta^2 J^2}{45}(1-q)^2 - \frac{1}{2}\beta J_0 m^2 + \langle\log 2\cosh\beta\hat{H}(z)\rangle_z. \tag{35}$$

The extremization condition reads

$$m = \langle\tanh\beta\hat{H}(z)\rangle_z, q = \langle\tanh^2\beta\hat{H}(z)\rangle_z. \tag{36}$$

The formula for m tells that the mean field is Gaussian distributed.

19 If the distribution of J is symmetric $J_0 = 0$) and $h = 0$, then $\hat{H}$ is odd, so $m = 0$. Therefore,

$$-\beta[f] = \frac{1}{4}\beta^2 J^2 (1-q)^2 + \langle \log 2\cosh \beta \hat{H}(z) \rangle_z. \tag{37}$$

Near the critical point q should be small, so

$$\beta[f] = -\frac{1}{4}\beta^2 J^2 - \log 2 - \frac{\beta^2 J^2}{4}(1 - \beta^2 J^2) q^2 + O[q^3]. \tag{38}$$

Thus, the spin-glass transition exists at $T_f = J/k_B$. However, free energy is not minimized. Therefore, we cannot discuss phase transition properly. The pathological nature of the result
21 can be seen from the negative entropy in the $T \to 0$ limit.

3. Replica Symmetry Breaking

24 To study the replica symmetry solution we expand the free energy around this solution and check the Hessian. The calculation is straightforward. The stability boundary is the *de*
27 *Almeida-Thouless line.*

Parisi solution:
This is beyond my understanding and taste, so I will ignore this topic.

TAP equation:
The local magnetization of the SK model satisfies the following *TAP equation* for the random coupling $\{J\}$:

$$m_i = \tanh \beta \left\{ \sum_j J_{ij} m_j + h_i - \beta \sum_j J_{ij}^2 (1 - m_j^2) m_i \right\}. \tag{39}$$

39 The third terms is called the *reaction field of Onsager* and is added to remove the effects of self-response: the magnetization m_i affects site j through internal field $J_{ij} m_i$ that changes m_j by the amount $\chi_{jj} J_{ij} m_i$,

$$\chi_{jj} = \frac{\partial m_j}{\partial h_j}_{h_j \to 0} = \beta(1 - m_j^2) \tag{40}$$

This increases the internal field at i be

$$J_{ij} \chi_{jj} J_{ij} m = \beta J_{ij}^2 (1 - m_j^2) m_i \tag{41}$$

40 The TAP equation can be obtained from the following free energy:

$$\begin{aligned} f_{\text{TAP}} &= -\frac{1}{2} \sum_{i \neq j} J_{ij} m_i m_j - \sum_i h_i m_i - \frac{\beta}{4} \sum_{i \neq j} J_{ij}^2 (1 - m_i^2)(1 - m_j^2), \\ &+ k_B T \sum_i \left\{ \frac{1 + m_i}{2} \log \frac{1 + m_i}{2} + \frac{1 - m_i}{2} \log \frac{1 - m_i}{2} \right\}. \end{aligned} \tag{42}$$

Here, the third term corresponds to the reaction field. This free energy may be derived from the SK Hamiltonian via *Plefka expansion.*

41 There is another method t derive the TAP equation—*cavity method.* The local magnetic field ($h = 0$) is $\tilde{h}_i = \sum_j J_{ij} S_j$. The energy may be written as

$$H = -\tilde{h}_i S_i - \sum{}' J_{kj} S_k S_j, \tag{43}$$

where $\sum'$ is the sum over the magnet without S_i. Since $\tilde{h}_i$ contains only S_j ($j \neq i$), the simultaneous distribution of S_i and $\tilde{h}_i$ may be written as

$$P(S_i, \tilde{h}_i) \propto e^{\beta \tilde{h}_i S_i} P(\tilde{h}_i), \tag{44}$$

where $P(\tilde{h}_i)$ is the distribution of $\tilde{h}_i$ for the magnet without S_i. $\tilde{h}_i$ is called the cavity field.
42 For the SK model, we may assume that the correlation between different sites are weak, so we may assume that $P(\tilde{h}_i)$ obeys $N(\langle h \rangle_i, V_i)$. $\langle \ \rangle_i$ implies the average over the magnet without S_i.

In this case, we obtain by a straightforward calculation

$$m_i = \tanh \beta \langle h \rangle_i \tag{45}$$

Thus, we need $\langle h \rangle_i$, the average of the field at i without S_i. If we write the true average of h_i as $\langle h \rangle$, then

$$\langle h \rangle = \langle h \rangle_i + V_i \langle S_i \rangle. \tag{46}$$

This can be obtained by the honest averaging of $\tilde{h}_i$ over $P(S_i, \tilde{h}_i)$. In other words,

$$\langle h \rangle_i = \sum_j J_{ij} m_j - V_i m_i. \tag{47}$$

Here,

$$V_i = \sum_{jk} J_{ij} J_{ik} (\langle S_j S_k \rangle_i - \langle S_j \rangle_i \langle S_k \rangle_i). \tag{48}$$

The off-diagonal terms cannot contribute due to the clustering property, so

$$V_i = \sum_j J_{ij}^2 (1 - \langle S_j \rangle_i^2) \simeq \sum_j J_{ij}^2 (1 - m_j^2). \tag{49}$$

Thus, we have recovered the TAP equation.

43 From the TAP equation RSB as well as RS results may be recovered.
44 The transition point may be obtained by the eigenvalue analysis of the random matrix $\{J\}$. It is expected that the TAP equation has very many solutions of $O[e^a N]$ ($a > 0$); only a fraction of the solutions correspond to minimum free energy solutions.

5. Gauge Theory of Spin Glasses

Consider the symmetric Edwards-Anderson model

$$H = -\sum_{\langle i,j \rangle} J_{ij} S)i S_j. \tag{50}$$

Here we asssume J_{ij} to be $\pm J$, The gauge transformation of spins is defined by

$$S_i \to S_i\sigma_i, \;\; J_{ij} \to J_{ij}\sigma_i\sigma_j. \tag{51}$$

H is invariant. Let p be the probabilty for $J_{ij} = J$. Then,

$$P(J_{ij}) = \frac{e^{K_p\tau_{ij}}}{2\cosh K_p}, \tag{52}$$

48 where $\tau_{ij} = \mathrm{sgn}(J_{ij})$, $J_{ij} = J\tau_{ij}$ and

$$e^{2K_p} = \frac{p}{1=p}. \tag{53}$$

The following argument applies to the coupling constant obeying the distribution of the form

$$P(J_{ij}) = P_0(|J_{ij}|)e^{aJ_{ij}}, \tag{54}$$

where a is a constant. The Gaussian model satisfies thei with $a = J_0/J^2$. The gaus transformation does not keep the distribution fucntion

$$P(J_{ij}) \to P(J_{ij})e^{aJ_{ij}-aJ_{ij}\sigma_i\sigma_j}. \tag{55}$$

Internal energy:

$$[E] = [\langle H\rangle] = \sum_\tau \frac{\exp\left(K_p\sum_{\langle ij\rangle}\tau_{ij}\right)}{(2\cosh K_p)^{N_B}} \frac{Tr_S\left(-J\sum_{\langle ij\rangle}\tau_{ij}S_iS_j\right)\exp\left(K\sum_{\langle ij\rangle}\tau_{ij}S_iS_j\right)}{Tr_S\exp\left(K\sum_{\langle ij\rangle}\tau_{ij}S_iS_j\right)}. \tag{56}$$

49 Applying the gauge transformation, we have

$$[E] = \sum_\tau \frac{\exp\left(K_p\sum_{\langle ij\rangle}\tau_{ij}\sigma_i\sigma_j\right)}{(2\cosh K_p)^{N_B}} \frac{Tr_S\left(-J\sum_{\langle ij\rangle}\tau_{ij}S_iS_j\right)\exp\left(K\sum_{\langle ij\rangle}\tau_{ij}S_iS_j\right)}{Tr_S\exp\left(K\sum_{\langle ij\rangle}\tau_{ij}S_iS_j\right)}. \tag{57}$$

This is invariant under the choice of $\{\sigma_i\}$. Therefore, we may average the above formula over all the choices of $\{\sigma_i\}$:

$$[E] = \frac{1}{2^N(2\cosh K_p)^{N_B}}\sum_\tau Tr_\sigma\exp\left(K_p\sum_{\langle ij\rangle}\tau_{ij}\sigma_i\sigma_j\right)\frac{Tr_S\left(-J\sum_{\langle ij\rangle}\tau_{ij}S_iS_j\right)\exp\left(K\sum_{\langle ij\rangle}\tau_{ij}S_iS_j\right)}{Tr_S\exp\left(K\sum_{\langle ij\rangle}\tau_{ij}S_iS_j\right)}. \tag{58}$$

If $K = K_p$ (*NIshimori line*), then

$$[E] = \frac{1}{2^N(2\cosh K_p)^{N_B}}\sum_\tau Tr_S\left(-J\sum_{\langle ij\rangle}\tau_{ij}S_iS_j\right)\exp\left(K\sum_{\langle ij\rangle}\tau_{ij}S_iS_j\right). \tag{59}$$

This can be calculated as

$$[E] = -N_BJ\tanh K. \tag{60}$$

This depends only on the total number N_B of bonds. Along the Nishimori line $[E]$ is non-
50 singular as a function of temperature. The line connects $T = J/K$ and $p = (1/2)(\tanh K_p +$
$1)$. $T = 0$, $p = 1$ (ferro) and the high temperature limite $T = \infty$, $p = 1/2$.

As can be guessed from (59) the boond energies are statistically independent on the N
51 line.

52 Upper bound of the specific heat can be estimated. The distributiojn of overlap q and
59 that of magnetization are identical..

67 Gauge glass (XY-model) is also discussed.

5. Error-Correcting Codes

78 Let $\xi_i = \pm 1$ and

$$J^0_{i_1 i_2, \cdots i_r} = \xi_{i_1} \cdots \xi_{i_r} \tag{61}$$

This is regarded as an input to a binary symmetric channel. The output of the chammel is $J_{i_1 i_2, \cdots i_r}$ and is equal to $\pm J^0_{i_1 i_2, \cdots i_r}$. The error probability is

$$P(J_{i_1 i_2, \cdots i_r} | J^0_{i_1 i_2, \cdots i_r}) = \frac{\exp(\beta_p J_{i_1 i_2, \cdots i_r} \xi_{i_1} \cdots \xi_{i_r})}{2 \cosh \beta_p}. \tag{62}$$

where β_p is determined as

$$e^{2\beta_p} = \frac{1-p}{p}. \tag{63}$$

Notice that p is the error probability:

$$p = \frac{1}{1 + e^{2\beta_p}} = \frac{e^{-\beta_p}}{2 \cosh \beta_p}, \; 1 - p = \frac{e^{\beta_p}}{2 \cosh \beta_p}. \tag{64}$$

79 Thus, the error probability reads

$$P(\boldsymbol{J} | \boldsymbol{\xi}) = \frac{1}{(2 \cosh \beta_p)^{N_B}} \exp\left(\beta_P \sum \boldsymbol{J} \cdot \boldsymbol{\xi}\right). \tag{65}$$

Here, the summation in $\boldsymbol{j} \cdot \boldsymbol{\xi} = \sum J_{i_1 i_2, \cdots i_r} \xi_{i_1 i_2, \cdots i_r}$ is take over all sets generated by (61) To decipher the corrupted code, we use Bayes' formula

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{P(B|A)P(A)}{\sum_A P(B|A)P(A)}. \tag{66}$$

Thus,

$$P(\boldsymbol{\sigma} | \boldsymbol{J}) = \frac{P(\boldsymbol{J} | \boldsymbol{\sigma}) P(\boldsymbol{\sigma})}{Tr_\sigma P(\boldsymbol{J} | \boldsymbol{\sigma}) P(\boldsymbol{\sigma})}. \tag{67}$$

If the message source produces all the message equally probably, we obtain

$$P(\boldsymbol{\sigma} | \boldsymbol{J}) = \frac{\exp(\beta_p \sum \boldsymbol{J} \cdot \boldsymbol{sigma})}{Tr_\sigma \exp(\beta_p \sum \boldsymbol{J} \cdot \boldsymbol{sigma})}. \tag{68}$$

This is nothing but the Boltzmann factor of an ising spin glass with randomly quenched interactions $\boldsymbol{J}$.

MAP decoding (maximum a posteriori probability) This maximizes $P(\boldsymbol{J}|\boldsymbol{\sigma})$ wrt to $\boldsymbol{\sigma}$. This is equivalent to maximizing $P(\boldsymbol{\sigma}|\boldsymbol{J})$ if $P(\boldsymbol{\sigma})$ is constant.

81 Another strategy is to study $P(\sigma_i|\boldsymbol{J})$.

$$= \hat{\xi}_i = \mathrm{sgn}(P(\sigma_i = 1|\boldsymbol{J}) - P(\sigma_i = -1|\boldsymbol{J})) \tag{69}$$

This is the estimate of decoded result and is called MPM (maximizer of posterior marginals). This is clearly different from MAP. MAP is equivalent of the low temeratureMPM. The above estimate may be understood as $\mathrm{sgn}(\langle\sigma_i\rangle)$.

The quality of decoding may be meaured by $\xi_i\hat{\xi}_i$ We compute its average

$$M(\beta) = Tr_\xi \sum_J P(\boldsymbol{\xi})P(\boldsymbol{J}|\boldsymbol{\xi})\xi_i\langle\sigma_i\rangle_\beta. \tag{70}$$

82 This is called the *overlap*. This is bounded by $M(\beta_p)$. To know the performance of the code, it is desirable that $M(\beta)$ may be estimated.

The infinite range model is solvable whose Hamiltonian is given by

$$H = -\sum_{i_1<\cdots<i_r} J_{i_1 i_2,\cdots i_r}\xi_{i_1}\cdots\xi_{i_r}. \tag{71}$$

84 The sum is over all possible combination of r spins taken from N spins. This can be solved
85 by the replica method. The replica symmetric solution is

$$q = q_{\alpha\beta} = \frac{1}{N}\sigma_i^\alpha\sigma_i^\beta,\ m = m_\alpha = \frac{1}{N}\sum\sigma_i^\alpha. \tag{72}$$

87 The equations governing q and m read

$$q = \langle\tanh^2\beta G\rangle, m = \langle\tanh\beta G\rangle, \tag{73}$$

with

$$G = J\sqrt{\frac{rq^{r-1}}{2}}u + j_0 r m^{r-1}. \tag{74}$$

Here $\langle\ \rangle$ is the average over u obeying $N(0,1)$.

We find

$$q = [\langle\sigma_i\rangle^2], m = [\langle\sigma_i\rangle]. \tag{75}$$

This suggests

$$[\langle\sigma_i\rangle^k] = \langle\tanh^k\beta G\rangle. \tag{76}$$

This is indeed correct, leading to

$$M(\beta) = [\mathrm{sgn}(\langle\sigma_i\rangle)] = \langle\mathrm{sgn}(G)\rangle. \tag{77}$$

88 The replica symmetry broken solution is also discussed..

95 Finite connectivity code.

102 Convolution code: